Spacetimes with Longitudinal and Angular Magnetic Fields in Third Order Lovelock Gravity

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Abstract

We obtain two new classes of magnetic brane solutions in third order Lovelock gravity. The first class of solutions yields an $(n+1)$-dimensional spacetime with a longitudinal magnetic field generated by a static source. We generalize this class of solutions to the case of spinning magnetic branes with one or more rotation parameters. These solutions have no curvature singularity and no horizons, but have a conic geometry. For the spinning brane, when one or more rotation parameters are nonzero, the brane has a net electric charge which is proportional to the magnitude of the rotation parameters, while the static brane has no net electric charge. The second class of solutions yields a spacetime with an angular magnetic field. These solutions have no curvature singularity, no horizon, and no conical singularity. Although the second class of solutions may be made electrically charged by a boost transformation, the transformed solutions do not present new spacetimes. Finally, we use the counterterm method in third order Lovelock gravity and compute the conserved quantities of these spacetimes.

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I. INTRODUCTION

In recent years asymptotically (anti)-de Sitter [(A)dS] spacetimes have attracted a great deal of attentions. While the interest on the asymptotically de Sitter spacetimes comes from the fact that at the present epoch the Universe expands with acceleration, the attention to asymptotically AdS spacetimes is due to the fact that there is a correspondence between supergravity (the low-energy limit of string theory) in $(n+1)$-dimensional asymptotically AdS spacetimes and conformal field theory (CFT) living on an n-dimensional boundary, known as the AdS/CFT correspondence [1]. The simplest way of having an asymptotically (A)dS spacetime is to add a cosmological constant term to the right hand side of Einstein equation. However, the cosmological constant meets its well known cosmological, fine tuning and coincidence problems [2]. In the context of classical theory of gravity, the second way of having an asymptotically (A)dS spacetime is to add higher curvature terms to the left hand side of Einstein equation. The way that we deal with the asymptotically (A)dS spacetime is the latter one. Indeed, it seems natural to reconsider the left hand side of Einstein equation, if one intends to investigate classical gravity in higher dimensions. The most natural extension of general relativity in higher dimensional spacetimes with the assumption of Einstein – that the left hand side of the field equations is the most general symmetric conserved tensor containing no more than two derivatives of the metric – is Lovelock theory. Lovelock [3] found the most general symmetric conserved tensor satisfying this property. The resultant tensor is nonlinear in the Riemann tensor and differs from the Einstein tensor only if the spacetime has more than 4 dimensions. Since the Lovelock tensor contains metric derivatives no higher than second order, the quantization of the linearized Lovelock theory is ghost-free [4]

The gravitational action satisfying the assumption of Einstein is precisely of the form proposed by Lovelock [3]:

$$I_G = \int d^dx\sqrt{-g}\sum_{k=0}^{[d/2]}\alpha_k\mathcal{L}_k, \tag{1}$$

where $[z]$ denotes integer part of z, α_k is an arbitrary constant and $\mathcal{L}_k$ is the Euler density of a $2k$-dimensional manifold,

$$\mathcal{L}_k = \frac{1}{2^k}\delta^{\mu_1\nu_1\cdots\mu_k\nu_k}_{\rho_1\sigma_1\cdots\rho_k\sigma_k}R_{\mu_1\nu_1}{}^{\rho_1\sigma_1}\cdots R_{\mu_k\nu_k}{}^{\rho_k\sigma_k} \tag{2}$$

In Eq. (2) $\delta^{\mu_1\nu_1\cdots\mu_k\nu_k}_{\rho_1\sigma_1\cdots\rho_k\sigma_k}$ is the generalized totally anti-symmetric Kronecker delta and $R_{\mu\nu}{}^{\rho\sigma}$ is the Riemann tensor. It is worthwhile to mention that in d dimensions, all terms for which $k > [d/2]$ are identically equal to zero, and the term $k = d/2$ is a topological term. So, only terms for which $k < d/2$ are contributing to the field equations. In this paper we want to restrict ourself to the first four terms of Lovelock gravity. The first term is the cosmological term, the second term is the Einstein term, and the third and fourth terms are the second order Lovelock (Gauss-Bonnet) and third order Lovelock terms, respectively. From a geometric point of view, the combination of these terms in seven-dimensional spacetimes, is the most general Lagrangian producing second order field equations, as in the four-dimensional gravity which the Einstein-Hilbert action is the most general Lagrangian producing second order field equations. Since the Lovelock Lagrangian appears in the low energy limit of string theory, there has in recent years been a renewed interest in Lovelock gravity. In particular, exact static spherically symmetric black hole solutions of the Gauss-Bonnet gravity (quadratic in the Riemann tensor) have been found in Ref. [5], and of the Maxwell-Gauss-Bonnet and Born-Infeld-Gauss-Bonnet models in Ref. [6]. The thermodynamics of the uncharged static spherically black hole solutions has been considered in [7], of solutions with nontrivial topology in [8, 9] and of charged solutions in [6, 10]. Recently NUT charged black hole solutions of Gauss-Bonnet gravity and Gauss-Bonnet-Maxwell gravity were obtained [11]. Not long ago one of us introduced two new classes of rotating solutions of second order Lovelock gravity and investigated their thermodynamics [12], and made the first attempt for finding exact static solutions in third order Lovelock gravity with the quartic terms [13], and presented the charged rotating black brane solutions of third order Lovelock gravity [14].

In this paper we are dealing with the issue of the spacetimes generated by spinning brane sources in $(n+1)$-dimensional third order Lovelock theory that are horizonless and have nontrivial external solutions. These kinds of solutions have been investigated by many authors in four dimensions. Static uncharged cylindrically symmetric solutions of Einstein gravity in four dimensions were considered in [15]. Similar static solutions in the context of cosmic string theory were found in [16]. All of these solutions [15, 16] are horizonless and have a conical geometry; they are everywhere flat except at the location of the line source. The extension to include the electromagnetic field has also been done [17, 18]. Some solutions of type IIB supergravity compactified on a four dimensional torus were considered in [19], which have no curvature singularity and no conic singularity.

The outline of our paper is as follows. We give a brief review of the field equations of third order Lovelock gravity in Sec. II. In Sec. III we first present a new class of static horizonless solutions which produce longitudinal magnetic field, and then generalize these solutions to the case of spacetimes with one or more rotation parameters. In Sec. IV we introduce those horizonless solutions that produce an angular magnetic field. Section V will be devoted to the use of the counterterm method to compute the conserved quantities of these spacetimes. We also compute the electric charge densities of the branes with rotation or boost parameters. We finish our paper with some concluding remarks.

II. FIELD EQUATIONS

The main fundamental assumptions in standard general relativity are the requirements of general covariance and that the field equations for the metric be second order. Based on the same principles, the Lovelock Lagrangian is the most general Lagrangian in classical gravity which produces second order field equations for the metric. The action of third order Lovelock gravity in the presence of electromagnetic field may be written as

$$\begin{aligned} I \;=\; & \int d^{n+1}x\sqrt{-g}\left(-2\Lambda+\mathcal{L}_1+\alpha_2\mathcal{L}_2+\alpha_3\mathcal{L}_3-F^2\right) \\ & +\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^nx\sqrt{-\gamma}\left\{L_{1b}+\alpha_2 L_{2b}+\alpha_3 L_{3b}\right\}, \end{aligned} \tag{3}$$

where Λ is the cosmological constant, α_2 and α_3 are Gauss-Bonnet and third order lovelock coefficients, $\mathcal{L}_1=R$ is just the Einstein-Hilbert Lagrangian, $\mathcal{L}_2=R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2$ is the Gauss-Bonnet Lagrangian and

$$\begin{aligned} \mathcal{L}_3 \;=\; & 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu}+8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa}+24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}{}_{\mu} \\ & +3RR^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu}+24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu}+16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu}-12RR^{\mu\nu}R_{\mu\nu}+R^3 \end{aligned} \tag{4}$$

is the third order Lovelock Lagrangian. In Eq. (3) $F^2=F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is electromagnetic tensor field and A_μ is the vector potential. The second integral in Eq. (3) is a boundary term which is chosen such that the variational principle is well defined [14, 20, 21]. In this integral $L_{1b}=K$, $L_{2b}=2(J-2\widehat{G}^{(1)}_{ab}K^{ab})$ and

$$\begin{aligned} L_{3b} \;=\; & 3(P-2\widehat{G}^{(2)}_{ab}K^{ab}-12\widehat{R}_{ab}J^{ab}+2\widehat{R}J \\ & -4K\widehat{R}_{abcd}K^{ac}K^{bd}-8\widehat{R}_{abcd}K^{ac}K^{b}_{e}K^{ed}), \end{aligned}$$

where $\gamma_{\mu\nu}$ is induced metric on the boundary, K is trace of extrinsic curvature of boundary, $\widehat{G}_{ab}^{(1)}$ and $\widehat{G}_{ab}^{(2)}$ are the n-dimensional Einstein and second order Lovelock tensors of the metric γ_{ab} [$\widehat{G}_{ab}^{(2)}$ will be given in Eq. (9)] and J and P are the trace of

$$J_{ab} = \frac{1}{3}(2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2K_{ab}) \tag{5}$$

and

$$\begin{aligned} P_{ab} \;=\; & \frac{1}{5}\{[K^4 - 6K^2K^{cd}K_{cd} + 8KK_{cd}K_e^dK^{ec} - 6K_{cd}K^{de}K_{ef}K^{fc} + 3(K_{cd}K^{cd})^2]K_{ab} \\ & -(4K^3 - 12KK_{ed}K^{ed} + 8K_{de}K_f^eK^{fd})K_{ac}K_b^c - 24KK_{ac}K^{cd}K_{de}K_b^e \\ & +(12K^2 - 12K_{ef}K^{ef})K_{ac}K^{cd}K_{db} + 24K_{ac}K^{cd}K_{de}K^{ef}K_{bf}\} \end{aligned} \tag{6}$$

Since in Lovelock gravity, only terms for which $k < d/2$ are contributing to the field equations and we want to consider the third order lovelock gravity, therefore we consider the d-dimensional spacetimes with $d \geq 7$. Varying the action (3) with respect to the metric tensor $g_{\mu\nu}$ and electromagnetic tensor field $F_{\mu\nu}$ the equations of gravitation and electromagnetic fields are obtained as:

$$G_{\mu\nu}^{(1)} + \Lambda g_{\mu\nu} + \alpha_2 G_{\mu\nu}^{(2)} + \alpha_3 G_{\mu\nu}^{(3)} = T_{\mu\nu}, \tag{7}$$

$$\nabla_\nu F^{\mu\nu} = 0, \tag{8}$$

where $T_{\mu\nu} = 2F^{\rho}{}_{\mu}F_{\rho\nu} - \frac{1}{2}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}$ is the energy-momentum tensor of electromagnetic field, $G_{\mu\nu}^{(1)}$ is just the Einstein tensor, and $G_{\mu\nu}^{(2)}$ and $G_{\mu\nu}^{(3)}$ are the second and third order Lovelock tensors given as [22]:

$$G_{\mu\nu}^{(2)} = 2(R_{\mu\sigma\kappa\tau}R_{\nu}{}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^{\sigma}{}_{\nu} + RR_{\mu\nu}) - \frac{1}{2}\mathcal{L}_2 g_{\mu\nu}, \tag{9}$$

$$\begin{aligned} G_{\mu\nu}^{(3)} \;=\; & -3(4R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\lambda\rho}R^{\lambda}{}_{\nu\tau\mu} - 8R^{\tau\rho}{}_{\lambda\sigma}R^{\sigma\kappa}{}_{\tau\mu}R^{\lambda}{}_{\nu\rho\kappa} + 2R_{\nu}{}^{\tau\sigma\kappa}R_{\sigma\kappa\lambda\rho}R^{\lambda\rho}{}_{\tau\mu} \\ & -R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\tau\rho}R_{\nu\mu} + 8R^{\tau}{}_{\nu\sigma\rho}R^{\sigma\kappa}{}_{\tau\mu}R^{\rho}{}_{\kappa} + 8R^{\sigma}{}_{\nu\tau\kappa}R^{\tau\rho}{}_{\sigma\mu}R^{\kappa}{}_{\rho} \\ & +4R_{\nu}{}^{\tau\sigma\kappa}R_{\sigma\kappa\mu\rho}R^{\rho}{}_{\tau} - 4R_{\nu}{}^{\tau\sigma\kappa}R_{\sigma\kappa\tau\rho}R^{\rho}{}_{\mu} + 4R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\tau\mu}R_{\nu\rho} + 2RR_{\nu}{}^{\kappa\tau\rho}R_{\tau\rho\kappa\mu} \\ & +8R^{\tau}{}_{\nu\mu\rho}R^{\rho}{}_{\sigma}R^{\sigma}{}_{\tau} - 8R^{\sigma}{}_{\nu\tau\rho}R^{\tau}{}_{\sigma}R^{\rho}_{\mu} - 8R^{\tau\rho}{}_{\sigma\mu}R^{\sigma}{}_{\tau}R_{\nu\rho} - 4RR^{\tau}{}_{\nu\mu\rho}R^{\rho}{}_{\tau} \\ & +4R^{\tau\rho}R_{\rho\tau}R_{\nu\mu} - 8R^{\tau}{}_{\nu}R_{\tau\rho}R^{\rho}{}_{\mu} + 4RR_{\nu\rho}R^{\rho}{}_{\mu} - R^2R_{\nu\mu}) - \frac{1}{2}\mathcal{L}_3 g_{\mu\nu} \end{aligned} \tag{10}$$

III. LONGITUDINAL MAGNETIC FIELD SOLUTIONS

Here, we want to obtain the $(n+1)$-dimensional solutions of Eqs. (7)-(8) which produce longitudinal magnetic fields in the Euclidean submanifold spans by the x^i coordinates ($i = 1, ..., n-2$). We assume that the metric has the following form:

$$ds^2 = -\frac{\rho^2}{l^2}dt^2 + \frac{d\rho^2}{f(\rho)} + l^2 f(\rho) d\phi^2 + \frac{\rho^2}{l^2}dX^2, \tag{11}$$

where $dX^2 = \sum_{i=1}^{n-2}(dx^i)^2$. Note that the coordinates x^i have the dimension of length, while the angular coordinate ϕ is dimensionless as usual and ranges in $0 \leq \phi < 2\pi$. The motivation for this metric gauge [$g_{tt} \propto -\rho^2$ and $(g_{\rho\rho})^{-1} \propto g_{\phi\phi}$] instead of the usual Schwarzschild gauge [$(g_{\rho\rho})^{-1} \propto g_{tt}$ and $g_{\phi\phi} \propto \rho^2$] comes from the fact that we are looking for a horizonless solution with conic singularity.

The gauge potential is given by

$$A_\mu = -\frac{2}{(n-2)}\frac{q l^{n-1}}{\rho^{n-2}}\delta_\mu^\phi \tag{12}$$

To find the function $f(\rho)$, one may use any components of Eq. (7). The simplest equation is the $\rho\rho$ component of these equations which can be written as

$$\begin{aligned}&\left[180\binom{n-1}{5}\alpha_3 \rho f^2 - 6\binom{n-1}{3}\alpha_2 f \rho^3 + \frac{n-1}{2}\rho^5\right] f' + \Lambda r^6 \\ &+360\binom{n-1}{6}\alpha_3 f^3 - 12\binom{n-1}{4}\alpha_2 \rho^2 f^2 + \binom{n-1}{2}\rho^4 f = 4q^2 l^{2n-4}\rho^{8-2n},\end{aligned} \tag{13}$$

where prime denotes the derivative with respect to ρ. The general solution of Eq. (13) is

$$f(\rho) = \frac{\rho^2}{b_3\alpha}\left\{\left(\sqrt{\gamma + k^2(\rho)} + k(\rho)\right)^{1/3} - \gamma^{1/3}\left(\sqrt{\gamma + k^2(\rho)} + k(\rho)\right)^{-1/3} + b_2\right\}, \tag{14}$$

where

$$\alpha_2 = b_2\frac{\alpha}{(n-2)(n-3)}, \qquad \alpha_3 = b_3\frac{\alpha^2}{72\binom{n-2}{4}}, \qquad \gamma = (b_3 - b_2^2)^3, \tag{15}$$

and

$$k(\rho) = -\frac{1}{2}b_2(3b_3 - 2b_2^2) + 3\alpha b_3^2\left(-\frac{\Lambda}{n(n-1)} + \frac{m}{\rho^n(n-1)} - \frac{4q^2}{(n-1)(n-2)\rho^{2n-2}l^{4-2n}}\right) \tag{16}$$

The constants m and q in Eq. (16) are the mass and charge parameters of the metric which are related to the mass and charge density of the solution. The function $k(\rho)$ approaches a constant as ρ goes to infinity, and the effective cosmological constant for the spacetime is

$$\Lambda_{\text{eff}} = -\frac{1}{\alpha b_3}\left\{b_2 + \left(\sqrt{\gamma + \lambda^2} + \lambda\right)^{1/3} - \gamma^{1/3}\left(\sqrt{\gamma + \lambda^2} + \lambda\right)^{-1/3}\right\},$$

where

$$\lambda = -\frac{1}{2}b_2(3b_3 - 2b_2^2) - \frac{3b_3^2\alpha\Lambda}{n(n-1)} \tag{17}$$

In the rest of the paper, we only investigate the case of $\gamma \geq 0$. In this case Λ_{eff} and $f(\rho)$ are real. The function $f(\rho)$ is negative for large values of ρ, if $\Lambda_{\text{eff}} > 0$. Since $g_{\rho\rho}$ and $g_{\phi\phi}$ are related by $f(\rho) = g_{\rho\rho}^{-1} = l^{-2}g_{\phi\phi}$, and therefore when $g_{\rho\rho}$ becomes negative (which occurs for large ρ) so does $g_{\phi\phi}$. This leads to an apparent change of signature of the metric from $(n-1)+$ to $(n-2)+$ as ρ goes to infinity, which is not allowed. Thus, Λ_{eff} should be negative which occurs provided

$$\lambda > -\frac{1}{2}b_2(3b_3 - 2b_2^2) \tag{18}$$

Equation (17) shows that the condition (18) is satisfied if $\Lambda < 0$.

In order to study the general structure of the solution given in Eq. (14), we first look for curvature singularities. It is easy to show that the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverge at $\rho = 0$ and therefore one might think that there is a curvature singularity located at $\rho = 0$. However, as we will see below, the spacetime will never achieve $\rho = 0$. The function $f(\rho)$ is negative for $\rho < r_+$ and positive for $\rho > r_+$, where r_+ is the largest root of $f(\rho) = 0$, which may be written as

$$(n-2)\rho^{2n-2}l^{4-2n}\Lambda - (n-2)\rho^{n-2}l^{4-2n}mn + 4q^2n = 0$$

Again $g_{\rho\rho}$ cannot be negative (which occurs for$\rho < r_+$), because of the change of signature of the metric from $(n-1)+$ to $(n-2)+$. Thus, one cannot extend the spacetime to $\rho < r_+$. To get rid of this incorrect extension, we introduce the new radial coordinate r as

$$r^2 = \rho^2 - r_+^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_+^2}dr^2$$

With this new coordinate, the metric (11) is

$$ds^2 = -\frac{r^2 + r_+^2}{l^2}dt^2 + \frac{r^2}{(r^2 + r_+^2)f(r)}dr^2 + l^2 f(r)d\phi^2 + \frac{r^2 + r_+^2}{l^2}dX^2, \tag{19}$$

where the coordinates r and ϕ assume the values $0 \leq r < \infty$ and $0 \leq \phi < 2\pi$, and $f(r)$ is now given as

$$f(r) = \frac{(r^2 + r_+^2)}{b_3\alpha}\left\{\left(\sqrt{\gamma + k^2(r)} + k(r)\right)^{1/3} - \gamma^{1/3}\left(\sqrt{\gamma + k^2(r)} + k(r)\right)^{-1/3} + b_2\right\}, \tag{20}$$

where

$$k(r) = -\frac{1}{2}b_2(3b_3-2b_2^2)+3\alpha b_3^2\left(-\frac{\Lambda}{n(n-1)}+\frac{m}{(r^2+r_+^2)^{n/2}(n-1)}-\frac{4q^2}{(n-1)(n-2)(r^2+r_+^2)^{(n-1)}l^{4-2n}}\right) \tag{21}$$

The gauge potential in the new coordinate is

$$A_\mu = -\frac{2}{(n-2)}\frac{ql^{(n-1)}}{(r^2+r_+^2)^{(n-2)/2}}\delta_\mu^\phi \tag{22}$$

The function $f(r)$ given in Eq. (20) is positive in the whole spacetime and is zero at $r=0$. Also note that the Kretschmann scalar does not diverge in the range $0 \le r < \infty$. Therefore this spacetime has no curvature singularities and no horizons. However, it has a conic geometry and has a conical singularity at $r=0$, since:

$$\lim_{r\to 0}\frac{1}{r}\sqrt{\frac{g_{\phi\phi}}{g_{rr}}} = lr_+G_0,$$

where

$$\begin{aligned} G_0 &= \frac{r_+^2}{3b_3^2\alpha}\left(\frac{k_0''}{\sqrt{4b_3-3b_2^2}}\right)\left\{b_2+\left(4[b_3\sqrt{4b_3-3b_2^2}-b_2(3b_3-2b_2^2)]\right)^{1/3}\right\}, \\ k_0'' &= k''(r=0) = -3\alpha b_3^2\left(\frac{mn}{(n-1)r_+^{n+2}}-\frac{8q^2}{(n-2)l^{4-2n}r_+^{2n}}\right), \end{aligned} \tag{23}$$

which is not equal to one. That is, as the radius r tends to zero, the limit of the ratio "circumference/radius" is not 2π and therefore the spacetime has a conical singularity at $r=0$.

Of course, one may ask for the completeness of the spacetime with $r \ge 0$ (or $\rho \ge r_+$). It is easy to see that the spacetime described by Eq. (19) is both null and timelike geodesically complete [18, 23]. In fact, we can show that every null or timelike geodesic starting from an arbitrary point can either extend to infinite values of the affine parameter along the geodesic or end on a singularity at $r=0$. Using the geodesic equation, one obtains

$$\dot{t} = \frac{l^2}{r^2+r_+^2}E, \quad \dot{x^i} = \frac{l^2}{r^2+r_+^2}P^i, \quad \dot{\phi} = \frac{1}{l^2 f(r)}L, \tag{24}$$

$$r^2\dot{r}^2 = (r^2+r_+^2)f(r)\left[\frac{l^2(E^2-\mathbf{P}^2)}{r^2+r_+^2}-\eta\right]-\frac{r^2+r_+^2}{l^2}L^2, \tag{25}$$

where the overdot denotes the derivative with respect to an affine parameter, and η is zero for null geodesics and $+1$ for timelike geodesics. E, L, and P^i's are the conserved quantities associated with the coordinates t, ϕ, and x^i, respectively, and $\mathbf{P}^2 = \sum_{i=1}^{n-2}(P^i)^2$. Notice that $f(r)$ is always positive for $r > 0$ and zero for $r = 0$. First we consider the null geodesics ($\eta = 0$). (i) If $E^2 > \mathbf{P}^2$ the spiraling particles ($L > 0$) coming from infinity have a turning point at $r_{tp} > 0$, while the nonspiraling particles ($L = 0$) have a turning point at $r_{tp} = 0$. (ii) If $E^2 = \mathbf{P}^2$ and $L = 0$, whatever the value of r, $\dot{r}$ and $\dot{\phi}$ vanish and therefore the null particles moves on the z-axis. (iii) For $E^2 = \mathbf{P}^2$ and $L \neq 0$, and also for $E^2 < \mathbf{P}^2$ and any values of L, there is no possible null geodesic. Second, we analyze the timelike geodesics ($\eta = +1$). Timelike geodesics is possible only if $l^2(E^2 - \mathbf{P}^2) > r_+^2$. In this case spiraling ($L \neq 0$) timelike particles are bound between r_{tp}^a and r_{tp}^b given by $0 < r_{tp}^a \leq r_{tp}^b < \sqrt{l^2(E^2 - \mathbf{P}^2) - r_+^2}$, while the turning points for the nonspiraling particles ($L = 0$) are $r_{tp}^1 = 0$ and $r_{tp}^2 = \sqrt{l^2(E^2 - \mathbf{P}^2) - r_+^2}$

A. Longitudinal magnetic field solutions with all rotation parameters

The rotation group in $n+1$ dimensions is $SO(n)$ and therefore the number of independent rotation parameters is $[n/2]$, where $[x]$ is the integer part of x. We now generalize the above solution given in Eq. (19) with $k \leq [n/2]$ rotation parameters. This generalized solution can be written as

$$\begin{aligned} ds^2 &= -\frac{r^2 + r_+^2}{l^2}\left(\Xi dt - \sum_{i=1}^{k} a_i d\phi^i\right)^2 + f(r)\left(\sqrt{\Xi^2 - 1}dt - \frac{\Xi}{\sqrt{\Xi^2 - 1}}\sum_{i=1}^{k} a_i d\phi^i\right)^2 \\ &\quad + \frac{r^2 dr^2}{(r^2 + r_+^2)f(r)} + \frac{r^2 + r_+^2}{l^2(\Xi^2 - 1)}\sum_{i<j}^{k}(a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2 + r_+^2}{l^2}dX^2, \end{aligned} \tag{26}$$

where $\Xi = \sqrt{1 + \sum_i^k a_i^2/l^2}$, dX^2 is the Euclidean metric on the $(n - k - 1)$-dimensional submanifold and $f(r)$ is the same as $f(r)$ given in Eq. (20). The gauge potential is

$$A_\mu = \frac{2}{(n-2)}\frac{ql^{(n-2)}}{(r^2 + r_+^2)^{(n-2)/2}}\left(\sqrt{\Xi^2 - 1}\delta_\mu^0 - \frac{\Xi}{\sqrt{\Xi^2 - 1}}a_i\delta_\mu^i\right); \quad \text{(no sum on i)} \tag{27}$$

Again this spacetime has no horizon and curvature singularity. However, it has a conical singularity at $r = 0$.

IV. ANGULAR MAGNETIC FIELD SOLUTIONS

In Sec. III we found a spacetime generated by a magnetic source which produces a longitudinal magnetic field along x^i coordinates. In this section we want to obtain a spacetime generated by a magnetic source that produce angular magnetic fields along the ϕ^i coordinates. Following the steps of Sec. III but now with the roles of ϕ and x interchanged, we can directly write the metric and vector potential satisfying the field equations (7)-(8) as

$$\begin{aligned} ds^2 &= -\frac{r^2+r_+^2}{l^2}dt^2 + \frac{r^2 dr^2}{(r^2+r_+^2)f(r)} \\ &\quad + (r^2+r_+^2)\sum_{i=1}^{n-2}(d\phi^i)^2 + f(r)dx^2, \end{aligned} \tag{28}$$

where $f(r)$ is given in Eq. (20). The angular coordinates ϕ^i's range in $0 \leq \phi^i < 2\pi$. The gauge potential is now given by

$$A_\mu = -\frac{2}{(n-2)}\frac{ql^{(n-2)}}{(r^2+r_+^2)^{(n-2)/2}}\delta_\mu^x \tag{29}$$

The Kretschmann scalar does not diverge for any r and therefore there is no curvature singularity. The spacetime (28) is also free of conic singularity. In addition, it is notable to mention that the radial geodesic passes through $r=0$ (which is free of singularity) from positive values to negative values of the coordinate r. This shows that the radial coordinate in Eq. (28) can take the values $-\infty < r < \infty$. This analysis may suggest that one is in the presence of a traversable wormhole with a throat of dimension r_+. However, in the vicinity of $r=0$, the metric (28) can be written as

$$\begin{aligned} ds^2 &= -\frac{r_+^2}{l^2}dt^2 + \frac{1}{r_+^2}G_0^{-1}dr^2 + r_+^2\sum_{i=1}^{n-2}(d\phi^i)^2 \\ &\quad + G_0 r^2 dx^2, \end{aligned} \tag{30}$$

where G_0 is given in Eq. (23). This clearly shows that, at $r=0$, the x direction collapses and therefore we have to abandon the wormhole interpretation.

To add linear momentum to the spacetime along the coordinate x^i, we perform the boost transformation

$$t \mapsto \Xi t - (v_i/l)x^i; \quad x^i \mapsto \Xi x^i - (v_i/l)t \quad \text{(no sum on } i\text{)}$$

in the $t - x_i$ plane, where v_i is a boost parameter and $\Xi = \sqrt{1 + \sum_i^{\kappa} v_i^2/l^2}$ (i can run from 1 to $\kappa \leq n-2$). One obtains

$$
\begin{aligned}
ds^2 \;=\; & -\frac{r^2+r_+^2}{l^2}R^2(r)\left(\Xi dt - l^{-1}\sum_{i=1}^{\kappa} v_i dx^i\right)^2 + f(r)\left(\sqrt{\Xi^2-1}dt - \frac{\Xi}{l\sqrt{\Xi^2-1}}\sum_{i=1}^{\kappa} v_i dx^i\right)^2 \\
& + \frac{r^2+r_+^2}{l^4(\Xi^2-1)}R^2(r)\sum_{i<j}^{\kappa}(v_i dx_j - v_j dx_i)^2 + \frac{r^2 dr^2}{(r^2+r_+^2)f(r)} + (r^2+r_+^2)R^2(r)d\Omega^2 \qquad (31)
\end{aligned}
$$

The gauge potential is given by

$$A_\mu = \frac{q b^{(3-n)\gamma}}{\Gamma(r^2+r_+^2)^{\Gamma/2}}\left(\sqrt{\Xi^2-1}\delta_\mu^t - \frac{\Xi}{l\sqrt{\Xi^2-1}}v_i\delta_\mu^i\right) \quad \text{(no sum on i)} \qquad (32)$$

This boost transformation is permitted globally since x^i is not an angular coordinate. Thus the boosted solution (31) is not a new solution. However, it generates an electric field.

V. CONSERVED QUANTITIES

In general, the action (3) is divergent when evaluated on solutions, as is the Hamiltonian and other associated conserved quantities. In Einstein gravity, one can remove the non logarithmic divergent terms in the action by adding a counterterm action $I_{\rm ct}$ which is a functional of the boundary curvature invariants [24]. The issue of determination of boundary counterterms with their coefficients for higher-order Lovelock theories is at this point an open question. However for the case of a boundary with zero curvature [$\widehat{R}_{abcd}(\gamma) = 0$], it is quite straightforward. This is because all curvature invariants are zero except for a constant, and so the only possible boundary counterterm is one proportional to the volume of the boundary regardless of the number of dimensions [14, 21]:

$$I_{\rm ct} = \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^n x\sqrt{-\gamma}\frac{n-1}{L}, \qquad (33)$$

where L is a scale length factor that depends on l, α_2 and α_3, that must reduce to l as α_2 and α_3 go to zero. Having the total finite action, one can use the quasilocal definition [25, 26] to construct a divergence free stress-energy tensor. For the case of manifolds with zero curvature boundary the finite stress energy tensor is

$$
\begin{aligned}
T^{ab} \;=\; & \frac{1}{8\pi}\{(K^{ab} - K\gamma^{ab}) + 2\alpha_2(3J^{ab} - J\gamma^{ab}) \\
& + 3\alpha_3(5P^{ab} - P\gamma^{ab}) + \frac{n-1}{L}\gamma^{ab}\ \} \qquad (34)
\end{aligned}
$$

The first three terms in Eq. (34) result from the variation of the surface terms in action (3) with respect to γ^{ab}, and the last term is the counterterm that is the variation of I_{ct} with respect to γ^{ab}. To compute the conserved charges of the spacetime, we choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in ADM form:

$$\gamma_{ab}dx^a dx^a = -N^2 dt^2 + \sigma_{ij}\left(d\phi^i + V^i dt\right)\left(d\phi^j + V^j dt\right), \tag{35}$$

where the coordinates ϕ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (34) can be written as

$$\mathcal{Q}(\xi) = \int_{\mathcal{B}} d^{n-1}\varphi\sqrt{\sigma}T_{ab}n^a\xi^b, \tag{36}$$

where σ is the determinant of the metric σ_{ij}, and n^a is the timelike unit normal vector to the boundary $\mathcal{B}$. In the context of counterterm method, the limit in which the boundary $\mathcal{B}$ becomes infinite ($\mathcal{B}_\infty$) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface $\mathcal{B}$ in to a reference of spacetime is required and the quantities which are computed are intrinsic to the spacetimes.

For our case, the magnetic solutions of third order Lovelock gravity, the first Killing vector is $\xi = \partial/\partial t$, and therefore its associated conserved charge is the total mass. Denoting the volume of the hypersurface boundary $\mathcal{B}$ at constant t and r by V_{n-1} the mass per unit volume V_{n-1} is

$$M = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}n^a\xi^b = \frac{1}{8\pi}ml^{p-1}\left[n(\Xi^2-1)+1\right], \tag{37}$$

where p is the number of angular coordinates ϕ^i of the spacetime. For the case of spacetimes with a longitudinal magnetic field, the charges associated with the rotational Killing symmetries generated by $\zeta_i = \partial/\partial\phi^i$ are the components of angular momentum per unit volume V_{n-1} of the system calculated as

$$J_i = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}n^a\zeta_i^b = \frac{1}{8\pi}n\Xi l^{p-1}ma_i \tag{38}$$

In the case of the spacetimes with an angular magnetic field introduced in Sec. IV, we encounter conserved quantities associated with translational Killing symmetries generated by $\varsigma_i = \partial/\partial x^i$. These conserved quantities are the components of linear momentum per unit volume V_{n-1} computed as

$$P_i = \int_{\mathcal{B}} d^{n-1}x\sqrt{\sigma}T_{ab}n^a\varsigma_i^b = \frac{1}{8\pi}n\Xi l^{p-2}mv_i \tag{39}$$

Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces for the spacetimes with a longitudinal magnetic field is

$$u^0 = \frac{1}{N}, \quad u^r = 0, \quad u^i = -\frac{N^i}{N},$$

and the electric field is $E^\mu = g^{\mu\rho}F_{\rho\nu}u^\nu$. Then the electric charge per unit volume V_{n-1} can be found by calculating the flux of the electromagnetic field at infinity, yielding

$$Q = \frac{V_{n-1}}{4\pi}ql^{p-1}\sqrt{\Xi^2 - 1} \tag{40}$$

Note that the electric charge is proportional to the magnitude of rotation parameters or boost parameters, and is zero for the case of a static solution.

VI. CLOSING REMARKS

In this paper, we added the second and third order Lovelock terms to the Einstein-Maxwell action with a negative cosmological constant. We introduced two classes of solutions which are asymptotically anti-de Sitter. The first class of solutions yields an $(n+1)$-dimensional spacetime with a longitudinal magnetic field [the only nonzero component of the vector potential is $A_\phi(r)$] generated by a static magnetic brane. We also generalized these solutions to the case of rotating spacetimes with a longitudinal magnetic field. We found that these solutions have no curvature singularity and no horizons, but have conic singularity at $r = 0$. In these spacetimes, when all the rotation parameters are zero (static case), the electric field vanishes, and therefore the brane has no net electric charge. For the spinning brane, when one or more rotation parameters are nonzero, the brane has a net electric charge density which is proportional to the magnitude of the rotation parameter given by $\sqrt{\Xi^2 - 1}$. The second class of solutions yields a spacetime with angular magnetic field. These solutions have no curvature singularity, no horizon, and no conic singularity. Again, we found that the branes in these spacetimes have no net electric charge when all the boost parameters are zero. We also showed that, for the case of traveling branes with nonzero boost parameters, the net electric charge density of the brane is proportional to the magnitude of the velocities of the brane ($\sqrt{\Xi^2 - 1}$).

The counterterm method inspired by the AdS/CFT correspondence conjecture has been widely applied to the case of Einstein gravity. Here we applied this method to the case of third order Lovelock gravity and calculated the conserved quantities of the two classes of solutions. We found that the counterterm has only one term, since the boundaries of our spacetimes are curvature-free. Other related problems such as the application of the counterterm method to the case of solutions of higher curvature gravity with nonzero curvature boundary remain to be carried out.

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